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Calculation of Potential Flow Around Airfoils Using a Discrete Vortex Method

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I. Abstract

A DISCRETE vortex method is presented for the calculation of the potential flow around arbitrary two-dimensional airfoils. The integral equations that arise when the zero internal velocity condition is used (to enforce the no-penetration condition) are discretized using the trapezium rule because of the exponential accuracy it exhibits in the case of an integrand with periodic odd derivatives. Careful cancellation is used for the singular integrand and higher order terms are derived that take into account the effect of the body contour curvature. The Kutta-Joukowski condition is introduced as an extra equation and the final overdetermined linear system of equations is solved using Gaussian elimination with partial pivoting to remove the redundancy, instead of dropping one equation as has been done in previous methods. The difficulties that arise at the trailing edge of the airfoil are treated by means of a transformation to a smooth contour. The scheme has proved to be very accurate when the calculations are compared with exact solutions and exhibits a high convergence rate. Moreover, for a given computational time the accuracy achieved by the present method is two to three orders of magnitude greater than that of previous schemes.

II. Contents

To calculate the two-dimensional, inviscid, incompressible flow around an airfoil in a uniform stream U_0, at an angle of incidence α, a continuous distribution of vorticity is placed on its surface. The unknown density $\gamma(s)$ of this distribution (where s is the arc length measured from the trailing edge) is equal to the local exterior surface velocity, and the condition that the body contour is a streamline is forced by requiring the velocity internal and tangential to it, to be zero.[1,2] This velocity can be expressed in terms of a line integral, which is a Cauchy principal value, around the circumference of the body. However, there are two major problems with the numerical calculation of this integral as we approach the trailing edge. One is due to integration errors since the two surfaces of the airfoil are very close together and the other because the integral is no longer a Cauchy principal value at the trailing edge. To overcome these difficulties, a transformation $\zeta = T(z)$ of the airfoil to a smooth contour is introduced, where z is the complex plane of the airfoil and ζ the transformed plane. The simplest transformation with the correct behavior is the Kármán-Trefftz with one of the singular points on the physical plane placed at the trailing edge and the other midway between the leading edge and the center of curvature there, as was suggested by Theodorsen.[3]

Then a parametrization $\gamma(s)\,\mathrm{d}s = G(\eta)\,\mathrm{d}\eta$ and $\zeta = \zeta(\eta) = T \times [z(\eta)]$ with $0 \le \eta \le 1$ gives us flexibility in the integration to follow. The parameter η is defined as $\eta = \sigma/\sigma_{\mathrm{tot}}$, σ_{tot} being the total length of the transformed contour and σ the arc length on it measured from the image of the trailing-edge point. The zero internal tangential velocity condition can now be written in the transformed plane as

$$Re\left\{ U_0 e^{-i\alpha} e^{i\phi_r} - \frac{G(\eta_r)}{2|\mathrm{d}\zeta/\mathrm{d}\eta|_r} + e^{i\phi_r}\frac{i}{2\pi}\int_0^1 \frac{G(\eta)\mathrm{d}\eta}{\zeta(\eta_r) - \zeta(\eta)} \right\} = 0 \tag{1}$$

where ϕ_r is the slope of the transformed contour at the point that corresponds to the value η_r of the parameter. A cancellation technique for the preceding integrand (Moore[4]) allows us to calculate the integral to accuracy higher than that achieved by the ordinary discrete vortex method. Then the preceding equation is discretized using the trapezium rule with $N = 2M$ pivotal points and so we finally have

$$\begin{aligned} Re\Bigg\{ &-\left[\frac{1}{2|\mathrm{d}\zeta/\mathrm{d}\eta|_r} + \frac{i}{2\pi} e^{i\phi_r}\,\Delta\eta \frac{(\mathrm{d}^2\zeta/\mathrm{d}\eta^2)_r}{2(\mathrm{d}\zeta/\mathrm{d}\eta)_r^2}\right] G(\eta_r) \\ &+ \frac{i}{2\pi} e^{i\phi_r}\Delta\eta \frac{(\mathrm{d}G/\mathrm{d}\eta)_r}{(\mathrm{d}\zeta/\mathrm{d}\eta)_r} - \frac{i}{2\pi} e^{i\phi_r}\Delta\eta {\sum_s}' \frac{G(\eta_s)}{\zeta(\eta_r) - \zeta(\eta_s)} \Bigg\} \\ &= -Re\{U_0 e^{-i\alpha} e^{i\phi_r}\} \end{aligned} \tag{2}$$

where the prime on the summation sign means that the value $s = r$ is omitted, $\Delta\eta = 1/N$ is the step length, and the derivatives that appear are approximated by finite differences. The periodicity of the odd derivatives of the integrand gives exponential accuracy to the preceding formula. The points equally spaced in η that are needed are generated with the use of a spline function.

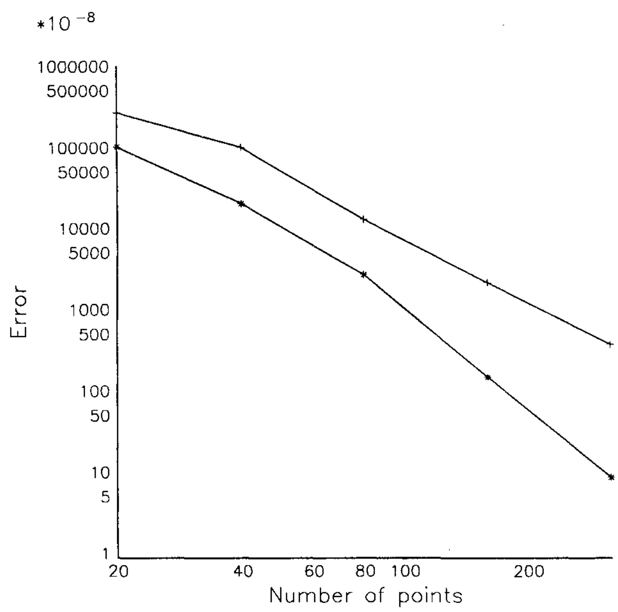

Fig. 1 Convergence of the method using second-order finite differences (+) and fourth-order finite differences (*) for the calculation of the derivatives (calculations for a Kármán-Trefftz profile).

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Table 1 Errors in the velocity and the pressure coefficient C_p in the case of a Mangler profile with a cusp at the trailing edge, when fourth-order finite differences were used

Number of points	$N = 20$	$N = 40$	$N = 80$	$N = 160$	$N = 320$
Maximum error for the velocity	2.979×10^{-3}	2.308×10^{-4}	1.134×10^{-5}	6.349×10^{-7}	4.551×10^{-8}
Mean error for the velocity	3.766×10^{-4}	1.943×10^{-5}	1.031×10^{-6}	5.104×10^{-8}	4.087×10^{-9}
Maximum error for C_p	4.345×10^{-3}	1.690×10^{-4}	8.308×10^{-6}	5.827×10^{-7}	6.454×10^{-8}
Mean error for C_p	6.384×10^{-4}	2.467×10^{-5}	1.531×10^{-6}	8.688×10^{-8}	7.473×10^{-9}

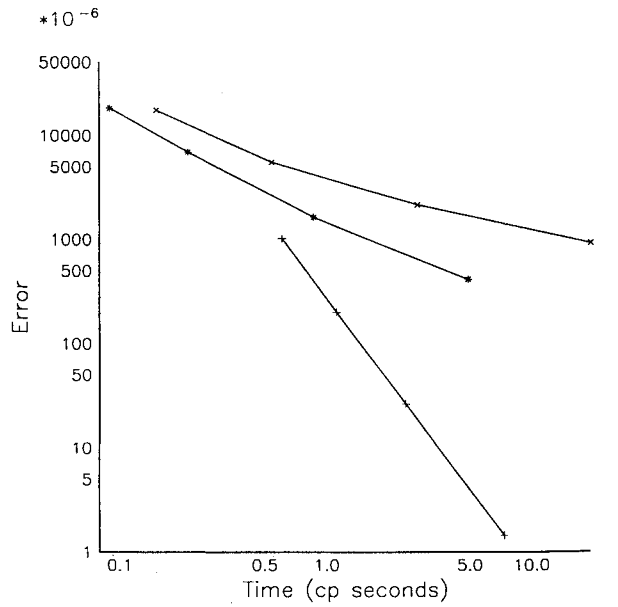

Fig. 2 Comparison of the accuracy achieved by the present method (+) and the methods by Smith and Hess[6] (×) and Kennedy and Marsden[7] (*) as a function of computational time (cp seconds on Cyber 960).

Equation (2) is forced at the N points and so gives a system of equations for the unknown vector G and thus for the unknown velocity on the airfoil. However, the Kutta condition is needed to prescribe the circulation of the potential flow, and it is introduced in the preceding system as an extra equation, stating that the value of G at the trailing edge point is zero. We now have a system of $N + 1$ equations with N unknowns, but the rank of its matrix is only N. For its solution Gaussian elimination with partial pivoting was used (Moore[5]), so that the redundancy is forced into the last row during the forward substitution stage. This approach is systematic and in general more accurate than randomly dropping one of the equations.

The method was checked for accuracy and convergence against exact solutions for some special profiles. The convergence rate is shown in Fig. 1 for the calculations of the flow around a Kármán-Trefftz profile (different from the transformation profile). The maximum convergence rate of the method is expected to be of order $(1/N)^4$ dictated by the cubic spline and the fourth-order finite differences. This maximum was not achieved until $N = 80$ pivotal points were used. The numerical values of the errors when the flow around a Mangler profile with a cusp at the trailing edge was considered are shown in Table 1. The calculations were extended over a wide range of profile thicknesses, trailing-edge angles, and angles of incidence, always exhibiting the same high accuracy and convergence. A comparison is attempted in Fig. 2 between the present scheme and the widely used methods of Smith and Hess[6] and Kennedy and Marsden.[7] It is obvious that the present method offers higher accuracy for the same computational cost.

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